

# Quantum dynamical theory for squeezing the output of a Bose-Einstein condensate

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## Abstract

A linear quantum dynamical theory for squeezing the output of the trapped Bose-Einstein condensate is presented with the Bogoliubov approximation. We observe that the non-classical properties, such as sub-Poisson distribution and quadrature squeezing effect, mutually oscillate between the quantum states of the applied optical field and the resulting atom laser beam with time. In particular, it is shown that an initially squeezed optical field will lead to squeezing in the outcoupled atomic beam at later times.

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## I. INTRODUCTION

Since the first observations of Bose-Einstein condensation in an atomic gas in 1995[1,2], there have been many interests in creating an atom laser and exploring its novel properties. In 1997[3], the MIT-group first realized a pulsed atom laser, by using a RF pulse to transfer the initially trapped condensate into the untrapped state. Later, successive experimental achievements in the design and amplification of atom laser[4-7] were obtained and stimulated amounts of theoretical works in both the output coupling and the properties of atom laser[8-10].

Recently, much attention were also paid to the problem of nonlinear atomic optics. Deng et al., for example, realized the four matter-wave mixing in their remarkable experiment by applying the optical technique of Bragg diffraction to the condensate[11]. Most recently, Moore et al. pointed out the possibility to realize the optical control on the quantum statistics of the output matter wave in the framework of nonlinear atomic optics[12].

The aim of this paper is to investigate the possibility to produce a squeezed atom laser from the trapped condensate, as the analogy of the squeezed light which is now available in laboratory[13]. The paper is organized as the following. In Sec. II, we present a model for squeezing the output of the trapped atoms with a many-boson system of two states, trapped state and untrapped state, with linear coupling. In the Bogoliubov approximation, the solutions of this system is also derived. Based on these solutions, the non-classical properties such as sub-Poisson distribution and quadrature squeezing effect are investigated in Sec. III, which leads to an interesting oscillation behavior of the quantum statistics between the coupling light field and the output atomic field. Finally, Sec. IV is a summary and outlook.

## II. MODEL AND SOLUTIONS

For simplicity, we shall assume that the atoms have two states,  $|1\rangle$  and  $|2\rangle$ , with the initial condensation occurring in the trapped state  $|1\rangle$ . State  $|2\rangle$ , which has different trapping properties and is typically unconfined by the magnetic trap, is coupled to  $|1\rangle$  by a one-mode squeezed optical field tuned near the  $|1\rangle \rightarrow |2\rangle$  transition. The interaction of the field may thus generate condensate in state  $|2\rangle$ , from an initial condensate which is entirely in state  $|1\rangle$ . The Hamiltonian of this dynamical model with linear coupling interaction can be written as[14] ( $\hbar = 1$ )

$$H = \omega_0 b_2^\dagger b_2 + \omega_a a^\dagger a + \omega'_R (ab_1 b_2^\dagger + a^\dagger b_1^\dagger b_2) \quad (1)$$

in terms of the creation and annihilation operators,  $b_1^\dagger$ ,  $b_2^\dagger$ ,  $b_1$  and  $b_2$ , of bosonic atoms for the magnetically trapped state  $|1\rangle$  and the untrapped state  $|2\rangle$  with level difference  $\omega_0$ ,  $a^\dagger$  and  $a$  is the creation and annihilation operators of the optical field with frequency  $\omega_a$ . Here  $\omega'_R = \sqrt{\omega_a/2\varepsilon_0 V}$ ,  $V$  is the effective mode volume and  $\varepsilon_0$  is the vacuum permittivity. The nonlinear interaction between the atoms and the quantized motion of atomic center of mass in the trapped state by an inhomogeneous magnetic field has been ignored, which is the main simplification of our model.

We suppose the initial state of the system is theoretically described as  $|\psi(0)\rangle = |\alpha\rangle_1 \otimes |\Phi(0)\rangle_s$  with  $|\Phi(0)\rangle_s = |0\rangle_2 \otimes |\xi\rangle$ . Here  $|\alpha\rangle_1$  is a Glauber coherent state of the operator  $b_1$  characterizing the condensed atoms in the trapped state  $|1\rangle$ , namely,  $b_1|\alpha\rangle = \sqrt{N_c}e^{-i\theta}|\alpha\rangle$ ;  $|0\rangle_2$  represents that the initial untrapped state  $|2\rangle$  is a vacuum state since there is no occupying atoms in it; and the initial state of the input optical field is the squeezed state[15]:  $|\xi\rangle = S(\xi)|m\rangle$ , where the squeezed operator  $S(\xi) = \exp[\xi(a^\dagger)^2 - \xi^*a^2]$  with  $\xi = \frac{1}{r}\exp(-2i\phi)$ , representing a unitary transformation on the coherent state  $|m\rangle$ . In the Bogoliubov approximation[16], we can ignore the slow change of the large number  $N_c$  of the condensed atoms in the trap, which means that the operators  $b_1$ ,  $b_1^\dagger$  can be replaced with a  $c$ -number  $\sqrt{N_c}$ . Therefore, the condensed component initially in a coherent state  $|\alpha\rangle_1$  remains in such a state while another component  $|\Phi(0)\rangle_s$  is governed by the Bogoliubov approximate Hamiltonian[14].

The Heisenberg equations about the operators  $b_2$  and  $a$  are

$$i\frac{\partial}{\partial t}b_2 = \omega_0 b_2 + \omega'_R a b_1, \quad i\frac{\partial}{\partial t}a = \omega_a a + \omega'_R b_1^\dagger b_2. \quad (2)$$

After taking the average value with respect to the coherent state  $|\alpha\rangle$ , the above equations can be written as

$$i\frac{\partial}{\partial t} \begin{pmatrix} b(t) \\ a(t) \end{pmatrix} = \begin{pmatrix} \omega_0 & \omega_R e^{-i\theta} \\ \omega_R e^{i\theta} & \omega_a \end{pmatrix} \begin{pmatrix} b(t) \\ a(t) \end{pmatrix}, \quad (3)$$

where  $\omega_R = \omega'_R \sqrt{N_c}$  and  $b_2$  was rewritten as  $b$ . By applying the technique of diagonalizing the coefficient matrix, we can obtain the exact solutions of the coupling equations as follows:

$$\begin{pmatrix} b(t) \\ a(t) \end{pmatrix} = \begin{pmatrix} \lambda_-(t) & -i\eta(t)e^{-i\theta} \\ -i\eta(t)e^{i\theta} & \lambda_+(t) \end{pmatrix} \begin{pmatrix} b(0) \\ a(0) \end{pmatrix} e^{-i\frac{1}{2}(\omega_0 + \omega_a)t}, \quad (4)$$

where  $\lambda_\pm(t) = \cos(I(\varphi)t) \pm i \sin(\varphi) \sin(I(\varphi)t)$ ,  $\eta(t) = \cos \varphi \sin(I(\varphi)t)$ , and  $\varphi, I(\varphi)$  are defined as  $\omega_0 - \omega_a = 2\omega_R \tan(\varphi)$ ,  $I(\varphi) = \omega_R / \cos(\varphi)$  and  $\varphi$  is an arbitrary constant. Now we consider the special case with a resonance frequency, namely,  $\omega_a = \omega_0 = \omega$  (or  $\varphi = 2n\pi$ ,  $n = \text{integer}$ ) and  $I(\varphi) = \omega_R$ , the eq.(4) can then be simplified as

$$\begin{pmatrix} b(t) \\ a(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega_R t) & -i \sin(\omega_R t) e^{-i\theta} \\ -i \sin(\omega_R t) e^{i\theta} & \cos(\omega_R t) \end{pmatrix} \begin{pmatrix} b(0) \\ a(0) \end{pmatrix} e^{-i\omega t}. \quad (5)$$

Obviously, for the evolution times satisfying  $\cos(\omega_R t_0) = 0$  or  $\omega_R t_0 = (n + 1/2)\pi$ , we have

$$b(t_0) = -i(-1)^n a(0) e^{-i(\omega t_0 + \theta)}, \quad a(t_0) = -i(-1)^n b(0) e^{-i(\omega t_0 - \theta)}, \quad (6)$$

where the parameter  $\theta$  will not affect our main results, as it can be seen below. In particular, eq.(6) leads to

$$\langle b^\dagger(t_0) b(t_0) \rangle = \langle a^\dagger(0) a(0) \rangle, \quad (7)$$

which indicates a complete *quantum conversion* of the statistical properties between the input photons and the output atoms.

### III. NON-CLASSICAL PROPERTIES OF THE OUTPUT ATOMIC FIELD

Using the solutions obtained above, we can calculate the average numbers and the fluctuations of the out-state photons as well as the output atoms. For the out-state photons, the final results are

$$\begin{aligned} \langle N_a(t) \rangle_s &= \langle \Phi(0) | a^\dagger(t) a(t) | \Phi(0) \rangle \\ &= \{m^* m \alpha_1 + [(m^*)^2 e^{-2i\phi} + m^2 e^{2i\phi}] \alpha_2 + \sinh^2 r\} \cos^2(\omega_R t), \end{aligned} \quad (8)$$

and

$$\begin{aligned} \langle \Delta N_a^2(t) \rangle_s &= \langle N_a^2(t) \rangle_s - \langle N_a(t) \rangle_s^2 \\ &= \{|m|^2 \alpha_1^2 + 2\alpha_2^2 (2|m|^2 + 1) + 2\alpha_1 \alpha_2 [(m^*)^2 e^{-2i\phi} + m^2 e^{2i\phi}]\} \cos^4(\omega_R t) \\ &\quad + \{\alpha_1 |m|^2 + \sinh^2 r + [(m^*)^2 e^{-2i\phi} + m^2 e^{2i\phi}] \alpha_2\} \sin^2(\omega_R t) \cos^2(\omega_R t), \end{aligned} \quad (9)$$

where  $\alpha_1 = \sinh^2 r + \cosh^2 r$  and  $\alpha_2 = \sinh r \cosh r$ . Similarly, the fluctuation of the output atoms is

$$\langle \Delta N_b^2(t) \rangle_s = \langle N_b^2(t) \rangle_s - \langle N_b(t) \rangle_s^2$$

$$\begin{aligned}
&= \{|m|^2\alpha_1^2 + 2\alpha_2^2(2|m|^2 + 1) + 2\alpha_1\alpha_2[(m^*)^2e^{-2i\phi} + m^2e^{2i\phi}]\} \sin^4(\omega_R t) \\
&+ \{\alpha_1|m|^2 + \sinh^2 r + [(m^*)^2e^{-2i\phi} + m^2e^{2i\phi}]\alpha_2\} \sin^2(\omega_R t) \cos^2(\omega_R t).
\end{aligned} \tag{10}$$

Obviously, if the squeezed angle is chosen as  $\phi = 0$  and  $m \in \mathbb{R}$  (real number), we then obtain

$$\begin{aligned}
&< \Delta N_a^2(t) >_s = [m^2(\alpha_1 + 2\alpha_2)^2 + 2\alpha_2^2] \cos^4(\omega_R t) \\
&+ [\sinh^2 r + (\alpha_1 + 2\alpha_2)m^2] \sin^2(\omega_R t) \cos^2(\omega_R t),
\end{aligned} \tag{11}$$

$$\begin{aligned}
&< \Delta N_b^2(t) >_s = [m^2(\alpha_1 + 2\alpha_2)^2 + 2\alpha_2^2] \sin^4(\omega_R t) \\
&+ [\sinh^2 r + (\alpha_1 + 2\alpha_2)m^2] \sin^2(\omega_R t) \cos^2(\omega_R t).
\end{aligned} \tag{12}$$

In particular, if the initial optical field is in a vacuum-squeezed state ( $m = 0$ ), the results can be written in the following simple form:

$$< N_a(t) >_s^0 = \sinh^2 r \cos^2(\omega_R t), \quad < N_a^2(t) >_s^0 = (2\alpha_2 + \sinh^4 r) \cos^4(\omega_R t), \tag{13}$$

and

$$< \Delta N_a^2(t) >_s^0 = \sqrt{2} \sinh r \cos^4(\omega_R t), \quad < \Delta N_b^2(t) >_s^0 = \sqrt{2} \sinh r \cosh r \sin^4(\omega_R t). \tag{14}$$

Now we suppose that the evolution times satisfy the following conditions

$$\cos(\omega_R t) = 0, \quad \text{or} \quad \omega_R t = (n + 1/2)\pi, \quad (n = 0, 1, 2, \dots) \tag{15}$$

then eqs.(11) and (12) become

$$< \Delta N_a^2(t) >_s^0 = 0, \quad < \Delta N_b^2(t) >_s^0 = m^2(\alpha_1 + 2\alpha_2)^2 + 2\alpha_2^2. \tag{16}$$

Obviously, even if we set  $m = 0$ , we have:  $< \Delta N_b(t) >_s^0 = \sqrt{2} \sinh r \cosh r \neq 0$ , which characterizes the existence of the squeezing effect for the output atomic field.

It is well-known that, to decide the statistical properties of a quantum field, we can define a  $Q$  parameter as[15]:

$$Q_a^s(t) = \frac{< \Delta N_a^2(t) >_s}{< N_a(t) >_s} - 1 \begin{cases} < 1 : & \text{Sub-Poisson distribution,} \\ = 0 : & \text{Poisson distribution,} \\ > 0 : & \text{Super-Poisson distribution.} \end{cases} \tag{17}$$

From eqs.(8), (11) and (12), the  $Q$  parameters of the out-state optical field and the output atomic field can be derived as ( $\phi = 0, m \in \mathbb{R}$ )

$$\begin{pmatrix} Q_a^s(t) \\ Q_b^s(t) \end{pmatrix} = \left[ \frac{m^2(\alpha_1 + 2\alpha_2)^2 + 2\alpha_2^2}{m^2(\alpha_1 + 2\alpha_2) + \sinh^2 r} - 1 \right] \begin{pmatrix} \cos^2(\omega_R t) \\ \sin^2(\omega_R t) \end{pmatrix}, \tag{18}$$

which yields an interesting result for  $m = 0$ :

$$\begin{pmatrix} Q_a^s(t) \\ Q_b^s(t) \end{pmatrix} = \alpha_1 \begin{pmatrix} \cos^2(\omega_R t) \\ \sin^2(\omega_R t) \end{pmatrix}. \tag{19}$$

In the initial state ( $t = 0$ ),  $Q_a^s(t) > 0$ ,  $Q_b^s(t) = 0$ , which means the initial state of the optical field is a squeezed state and the initial state of the atom field is a vacuum state, as they should be. When the evolution time  $t_0$  satisfies  $\cos(\omega_R t_0) = 0$ , then  $Q_a^s(t) = 0$ ,  $Q_b^s(t) > 0$ , which means the initial squeezed optical field transforms into a coherent state as well as the initial coherent atom field is now squeezed. It is this interesting periodically oscillating behavior which holds the promise to be observed in the laboratory.

At last, we proceed to calculate the quadrature squeezing of output atomic field. According to Ref.[15], the field quadratures  $X_{1a}$ ,  $X_{1b}$ ,  $X_{2a}$  and  $X_{2b}$  are defined as

$$X_{1a} = \frac{1}{2}(a + a^\dagger), \quad X_{2a} = \frac{1}{2i}(a - a^\dagger), \quad X_{1b} = \frac{1}{2}(b + b^\dagger), \quad X_{2b} = \frac{1}{2i}(b - b^\dagger). \tag{20}$$

Following Bruzek et al.[17], we introduce the squeezed coefficients

$$S_i = \frac{\langle (\Delta X_i)^2 \rangle - \frac{1}{2} | \langle [X_1, X_2] \rangle |}{\frac{1}{2} | \langle [X_1, X_2] \rangle |}, \quad i = 1, 2 \quad (21)$$

one then gets

$$\begin{aligned} S_{1b}(t) &= 2 \langle N_b(t) \rangle + 2 \operatorname{Re} \langle b^2(t) \rangle - (\operatorname{Re} \langle b(t) \rangle)^2, \\ S_{2b}(t) &= 2 \langle N_b(t) \rangle - 2 \operatorname{Re} \langle b^2(t) \rangle - (\operatorname{Im} \langle b(t) \rangle)^2. \end{aligned} \quad (22)$$

On account of ( $\phi = 0, m = 0$ )

$$\langle b^2(t) \rangle_s = -\sinh r \cosh r e^{-2i\omega t} \sin^2(\omega_R t), \quad \langle b^\dagger b(t) \rangle_s = \sinh^2 r \cosh^2 r \sin^2(\omega_R t), \quad (23)$$

and  $\langle b(t) \rangle_s = 0$ , we can finally obtain

$$\begin{aligned} S_{1b}(t) &= 2 \sinh r \{ \sinh r - \cosh r \cos[2(\omega t + \theta)] \} \sin^2(\omega_R t), \\ S_{2b}(t) &= 2 \sinh r \{ \sinh r + \cosh r \cos[2(\omega t + \theta)] \} \sin^2(\omega_R t). \end{aligned} \quad (24)$$

Obviously, for the initial state of the atomic field, eq.(24) yields  $S_{1b}(0) = S_{2b}(0) = 0$ , which means there is no squeezing, as it should be.

After a period of evolution time, namely,  $\omega t + \theta = n\pi$ , but  $\omega_R t \neq n\pi$ , we can get the following results for the output atomic field

$$S_{1b}(t) = -2 \sinh r e^{-r} \sin^2(\omega_R t) < 0, \quad S_{2b}(t) = 2 \sinh r e^{-r} \sin^2(\omega_R t) > 0, \quad (25)$$

which just means that the quadrature component  $X_{1b}$  is squeezed.

However, after another period of evolution time, namely,  $\omega t + \theta = (n + 1/2)\pi$ , but  $\omega_R t \neq n\pi$ , it will become

$$S_{1b}(t) = 2 \sinh r e^{-r} \sin^2(\omega_R t) > 0, \quad S_{2b}(t) = -2 \sinh r e^{-r} \sin^2(\omega_R t) < 0, \quad (26)$$

which means that the squeezing effect transfers to  $X_{2b}$  component. In the same way, we can also calculate the squeezed coefficients for the optical field, which shows a similar behavior as the atomic field.

#### IV. CONCLUSION AND OUTLOOK

In this paper, we have theoretically presented a model for squeezing the output of the trapped condensed atoms with a many-boson system of two states with linear coupling. In the Bogoliubov approximation, its solutions for the many-body problem show that the non-classical properties, such as sub-Poisson distribution and quadrature squeezing effect, mutually oscillate between the quantum states of the applied optical field and the resulting atom laser beam with time. Hence after some period of coupling interaction, the initially squeezed light will transform into a coherent state while the output atomic field is in a squeezed state, which means a squeezed output of atomic beam in the propagating mode.

The availability of a squeezed atom laser would certainly be useful in future applications of cold atoms and our investigation provides a beginning point towards its realization. However, our model does not include atom-atom interactions and therefore only works when the atomic field is very dilute. Future work will involve generalizing this squeezed atom laser model to include the influence of atomic interactions, the effects of the trapping field and the shape of the condensate. Furthermore, it is also important to analyze in detail the requirements for the actual experimental realization (which would probably require an optical cavity) and the conditions for the relevant parameters, which may deserve further works in the future.

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